The Use of Similarity in Old Babylonian Mathematics

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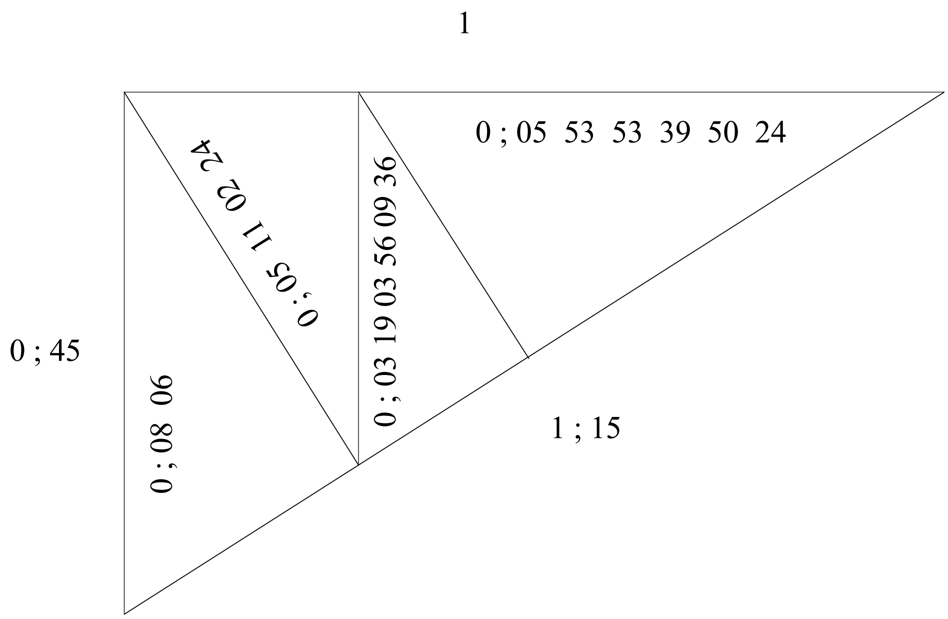


Diagram #1

The above diagram is a representation and transliteration of the ancient Babylonian tablet, **IM 55357** from Tell Harmal. IM 55357, which was likely written during the First Babylonian Empire (1900-1600 B.C.E.), has been used by historians of mathematics to prove that the Babylonians understood similarity. On what grounds can they make this claim? Is their assertion valid? Let us examine the tablet.

In Eleanor Robson's translation, the problem is stated as follows:

> A wedge. The length is 1, the long length 1;15, the upper width 0;45, the complete area 0;22 30. Within 0;22 30, the complete area, the upper area is 0;08 06, the next area 0;05 11 02 24, the third area 0;03 19 03 56 09 36, the lower area 0;05 53 53 39 50 24. What are the upper length, the middle length, the lower length, and the vertical? [**10**]

Since the Babylonians used a base 60 number system, 12 20 could represent $12(60)+20=740$ or $12+\frac{20}{60}=12.\overline{3}$, among other possibilities. The Babylonians did not use decimal places in their mathematics because the results would still be consistent no matter where the decimal place happened to fall. For now, we shall insert decimal places, indicated by semicolons, arbitrarily.

Converting the numbers to base 10, we can rewrite the problem as:

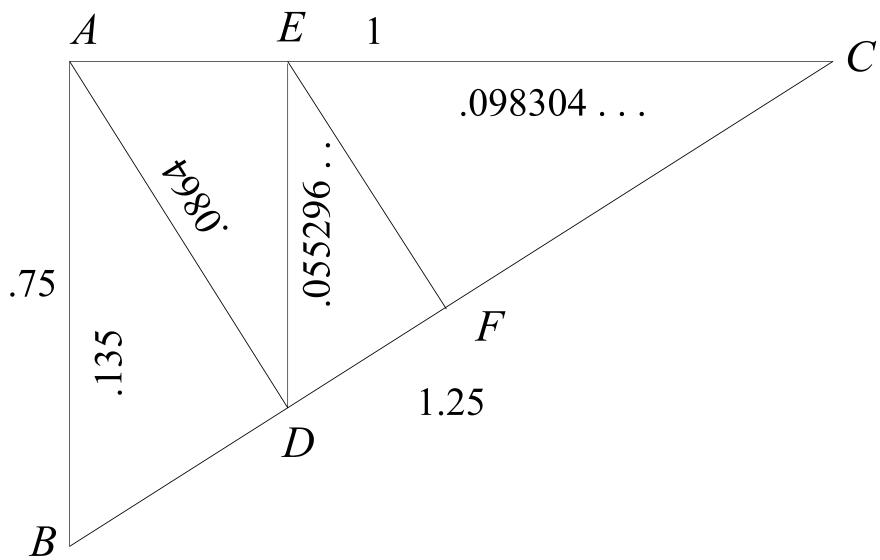


Diagram #2

A wedge. The length is 1, the long length 1.25, the upper width .75, the complete area .375. Within .375, the complete area, the upper area is .135, the next area .0864, the third area .055296…, the lower area .098304... What are the upper length, the middle length, the lower length, and the vertical?

Our next task is to interpret the "upper length," "the upper area," and these other ancient descriptions of sides and regions.

Since the areas are labeled on the diagram, we can interpret "the complete area" to be the area of ΔABC. The "upper area" corresponds to ΔABD. The "next area" is the area of ΔADE. The "third area" is the area of ΔEDF, while the "lower area" is the area of ΔEFC. "The length," "the long length," and "the upper width" could be *AC*, *BC*, and *AB* respectively.

What exactly is the problem asking? What are "the upper length," "the middle length," "the lower length," and "the vertical"? We cannot be certain, as different words are used to identify the lengths in the solution than the words used in the problem. (Because of these differing words, we cannot even be sure when the Babylonians decided that they reached a solution.)

While we may not be able to reach a consensus on the lengths which the problem requests, we do note that the tablet provides solutions for *BD*, *AD*, and *AE*. It would presumably continue to the next leg of ΔAED, *ED*, but the scribe never finished. For now, let us set aside which lengths the tablet asked for explicitly. If we deal exclusively in terms of the answers found and the methods employed, we can phrase the problem like this:

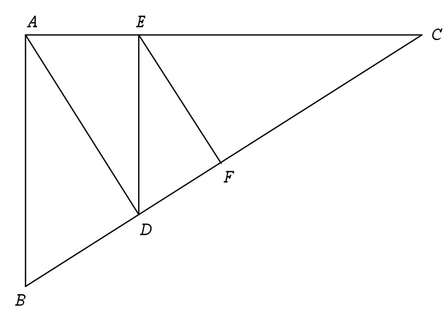


Diagram #3

In ΔABC, $AB = .75$, $AC = 1$, and $BC = 1.25$. The area of ΔABC is .375, the area of ΔABD is .135, the area of ΔADE is .0864, the area of ΔEDF is .055296…, and the area of ΔEFC is .098304… Solve for *BD*, *AD*, *AE*, and *ED*. (For more information on which lengths exactly the problem is solving for, the reader is invited to consult the Appendix.)

Right-angled similar triangles leap out at the reader, but since none of the angles are labeled, can we distinguish between a right angle and one of 89.75 degrees? As we will observe later, the Babylonian method for finding area did not involve multiplying an altitude by the base of the triangle, but rather multiplying together the two legs of a triangle. In order for their method to work, they had to have been dealing with right triangles. To what extent they recognized the triangles as "right" is an entirely different question. Perhaps they developed theorems for lines meeting in a T-shaped, perpendicular formation. They may have identified right triangles as half-rectangles. In any case, since the Babylonian solution holds true only for right-angled triangles, we shall assume, as many scholars have done, that $\angle BAC, \angle BDA, \angle DEA$, and $\angle EFD$ are right angles.

Armed with this information, we can now solve the problem by using simultaneous equations. We can solve for *BD* and *AD*, the legs of "the upper triangle," with two sets of equations based on similarity and area:

$$\Delta ABC \sim \Delta DBA$$

$$\text{Therefore, } \frac{AB}{AC} = \frac{BD}{AD}$$

$$\frac{3}{4} = \frac{BD}{AD}$$

$$AD = \frac{4}{3} BD \tag{1}$$

$$a(ABD) = \frac{1}{2}(BD)(AD)$$

$$\frac{1}{2}(BD)(AD) = .135 \tag{2}$$

We can solve for *BD* by substituting (1) into (2):

$$\frac{2}{3} BD^2 = .135$$

$$BD = .45$$

Substituting $BD = .45$ into equation (1), we can obtain: $AD = .6$.

Did the ancient Babylonians use such methods? Did they understand the concept of similar triangles? If so, to what extent did they understand it? Did they use similarity to show that the sides of two triangles are proportional? Did they realize that the ratio between the areas of two similar triangles is equal to the ratio of the sides squared? Did they know that area equals $\frac{1}{2}bh$? Did they use direct substitution?

To find out, let us examine their solution to IM 55357. On the next two pages, I have presented two translations in a two-column format, one by Eleanor Robson, professor at the University of Cambridge, and the other by Jens Høyurp, author of *Lengths, Widths, and Surfaces* (2002) and professor at the University of Roskilde in Denmark. The translation is on the left hand side, while the modern equivalent can be found in the right column.

Robson

You, when you proceed: solve the reciprocal of 1, the length.

$$\frac{1}{AC}=\frac{1}{1}=1$$

Multiply by 0;45. You will see 0;45.

$$\frac{AB}{AC}=\frac{3}{4}$$

Multiply 0;45 by 2. You will see 1;30.

$$2\frac{AB}{AC}=\frac{3}{2}$$

Multiply 1;30 by 0;08 06, the upper area. You will see 0;12 09

$$\left(2\frac{AB}{AC}\right)\left(a(\Delta ABD)\right)=\frac{3}{2}(.135)=.2025$$

What squares 0;12 09? 0;27, the [length of the upper] wedge, squares (it).

$$\sqrt{\left(2\frac{AB}{AC}\right)\left(a(\Delta ABD)\right)}=\sqrt{.2025}=.45=BD$$

Break off [half of] 0;27. You will see 0;13 30.

$$\frac{BD}{2}=.225$$

Solve the reciprocal of 13;30.

$$\frac{2}{BD}=\frac{1}{.225}$$

Multiply by 0;08 06, the upper area. You will see 0;36, the dividing (?) length of 0;45, the width.

$$\left(\frac{2}{BD}\right)\left(a(\Delta ABD)\right)=\left(\frac{1}{.225}\right)(.135)=.6=AD$$

Turn back. Take away 0;27, the length of the upper wedge from 1;15. You leave behind 0;48.

$$(BC-BD)=(1.25-.45)=.8=DC$$

Solve the reciprocal of 0;48. You will see 1;15.

$$\frac{1}{DC}=\frac{1}{.8}=1.25$$

Multiply 1;15 by 0;36. You will see 0;45.

$$\frac{AD}{DC}=1.25(.6)=.75$$

Multiply 0;45 by 2. You will see 1;30.

$$2\frac{AD}{DC}=2(.75)=1.5$$

Multiply 1;30 by 0;05 11 02 24. You will see 0;07 46 33 36.

$$\left(2\frac{AD}{DC}\right)\left(a(\Delta ADE)\right)=(1.5)(.0864)=.1296$$

What squares 0;07 46 33 36? 0;21 36 squares it.

$$\sqrt{\left(2\frac{AD}{DC}\right)\left(a(\Delta ADE)\right)}=\sqrt{.1296}=.36$$

The width of the second triangle is 0;21 36.

$$AE=.36$$

Break off half of 0; 21 36. You will see 0;10 48.

$$\frac{AE}{2}=\frac{.36}{2}=.18$$

Solve the reciprocal of 0;10 48.

$$\frac{2}{AE}=\frac{1}{.18}$$

Multiply by…

The scribe did not finish the text.

[10]

Høyrup

You, to know the proceeding, igi 1, the length detach, to 0;45 raise

$$\frac{1}{AC} = \frac{1}{1} = 1 \rightarrow \frac{AB}{AC} = (1)\left(\frac{3}{4}\right)$$

0;45 you see. 0;45 to 2 raise. 1;30 you see, to 0;08 06, the upper surface

$$2\frac{AB}{AC} = \frac{3}{2} \rightarrow \left(2\frac{AB}{AC}\right)\big(a(\Delta ABD)\big) = \frac{3}{2}(.135)$$

Raise, 0;12 9 you see. (By) 0;12 9, what is equalside? 0;27 is equalside.

$$\frac{3}{2}(.135) = .2025 \rightarrow \sqrt{.2025} = .45 = BD$$

0;27 the width, 0;27 break, 0;13 30 you see. Igi 0;13 30 detach,

$$\frac{BD}{2} = \frac{.45}{2} = .225 \rightarrow \frac{2}{BD} = \frac{1}{.225}$$

To 0;08 06 the upper [surf]ace raise, 0;36 you see, the length (which is) the counterpart of the length 0;45, the width.

$$\left(\frac{2}{BD}\right)\big(a(\Delta ABD)\big) = \left(\frac{1}{.225}\right)\big(.135\big) = .6 = AD$$

Turn around. The length 0;27, of the upper triangle, from 1;15 tear out,

$$(BC - BD) = (1.25 - .45) = .8 = DC$$

48 leave. Igi 48 detach, 1;15 you see, 1;15 to 0;36 raise.

$$\frac{1}{DC} = \frac{1}{.8} = 1.25 \rightarrow \frac{AD}{DC} = 1.25(.6)$$

0;45 you see. 0;45 to 2 raise. 1;30 you see, to 0;05 11 02 24 raise,

$$1.25(6)=.75 \rightarrow 2\frac{AD}{DC} = 2(.75) = 1.5$$

0;07 46 33 36 you see. (by) 0;07 46 33 36, what is equalside?

$$\left(2\frac{AD}{DC}\right)\big(a(\Delta ADE)\big) = (1.5)(.0864) = .1296 \rightarrow$$

$$\sqrt{.1296} = ?$$

0;21 36 is equal side, 0;21 36 the width of the 2nd ‹tria›ngle,

$$\sqrt{.1296} = .36 = AE$$

The moiety of 0;21 36 break, 0;10 48 you see. Igi 0;10 48 detach.

$$\frac{AE}{2} = \frac{.36}{2} = .18 \rightarrow \frac{2}{AE} = \frac{1}{.18}$$

To …

The scribe did not finish the text

[**7**]

These two translations are pronouncedly different from one another. For instance, Robson translates the first step as "solve the reciprocal of 1, the length," while Høyrup writes, "You, to know the proceeding, igi 1, the length detach." In his translation, Høyrup intends to remain as close to the *intent* of the actual text as possible. Thus, instead of writing "solve the reciprocal," he uses the Babylonian word for reciprocal, "igi." The word "detach" indicates that we are focusing on one side of a triangle in order to perform mathematical transformations.

In the second step, Robson writes "Multiply by 0;45. You will see 0;45," while Høyrup transcribes "to 0;45 raise; 0;45 you see." We can see that Høyrup is attempting to give us a more accurate picture, even down to the Babylonian use of the object-subject-verb formation in "0;45 you see." Høyrup is continually mindful of the geometrical underpinnings. He uses the term "raise" instead of multiply, because he views Babylonian multiplication to have a connotation of scaling or "raising" a side of a geometric figure. He also underscores the fact that Babylonians did not divide, but only multiplied by reciprocals.

We see the same terminology in the third as well as the fourth step. Robson translates the third step using the word "multiply": "Multiply 0;45 by 2. You will see 1;30," while Høyrup interprets it with "raise": "0;45 to 2 raise. 1;30 you see…" Likewise, in the fourth step, Robson transcribes, "Multiply 1;30 by 0;08 06, the upper area. You will see 0;12 09," while Høyrup writes, "Raise, 0;12 9 you see."

In the fifth step, Robson translates, "What squares 0;12 09? 0;27, the [length of the upper] wedge squares (it)," while Høyrup records, "(by) 0;12 9, what is equalside? 0;27 is equalside." Høyrup believes that when Babylonians took the square root of a number *k*, they were actually finding the side of a square with area *k*. Consequently, they would refer to a square root by the same terminology with which they referred to a square, "equalside."

As both Høyrup and Robson note in the sixth step, instead of taking one-half of something, the Babylonians use the expression "Break off 0;27" or "0;27 break." In the seventh step, we see the same "reciprocal"/"igi-detach" terminology employed by the two different translators.

The first part of the eighth step, as translated by Robson is, "Multiply by 0;08 06, the upper area," while Høyrup sets down, "To 0;08 06, the upper [surf]ace raise…" Høyrup believes that the Babylonians understood area in a different sense than we did and thus uses the term "surface."

In the second part, Robson's translation has, "You will see 0;36, the dividing (?) length of 0;45, the width," with Høyrup's as, "0;36, you see, the length (which is) the counterpart of the length 0;45, the width." Høyrup justifies his belief in Babylonian similarity with this phrase, "counterpart," while Robson, as evidenced by her question mark, appears unsure as to the true interpretation of this word, although "dividing length," which is quite different from "counterpart," is her best interpretation.

"Turn back" is the next line of the tablet, although Høyrup translates it as "turn yourself around." Høyrup implies that the line is simply a transition to a new section. However, I believe that the idea of turning backward or around may indicate the scribe's identification of the recursive nature of his method. First, using the ratio of the bases of ΔABC and the area of subtriangle ΔABD, the scribe finds the subtriangle's bases, *BD* and *AD*. He repeats the process to find *AE*, but now he uses the ratio of the bases of ΔADC and area of subtriangle ΔADE. Although he did not finish the tablet, he seemed to be repeating an algorithm. The Babylonians seized upon an easily repeatable method that would enable them to find the sides of successive subtriangles within a large triangle. Perhaps the tablet went unfinished because the scribe felt that the method had been sufficiently illustrated for the reader to continue on his own.

Our understanding of IM55357 hinges upon two recursive procedures, the first of which we produce as the formula $\sqrt{\left(2\frac{AB}{AC}\right)(a(\Delta ABD))} = BD$. This procedure is used twice, and in its second iteration, it yields $\sqrt{\left(2\frac{AD}{DC}\right)(a(\Delta ADE))} = AE$. How did the Babylonians arrive at this procedure? Writing in the 2000 edition of *Crest of the Peacock*, George Gheverdese Joseph, professor at the Universities of Exeter and Manchester, contends that to produce this formula, the Babylonians employed *similarity*. He implies that they would have used the following equations to derive their method:

$$\text{First, since } \Delta ABC \sim \Delta DBA, \frac{AB}{AC} = \frac{BD}{AD}.$$

$$\text{Second, } a(ABD) = \frac{1}{2}(BD)(AD)$$

Indeed, these equations can be used to justify the method represented by the above formula, as:

$$\sqrt{\left(2\frac{AB}{AC}\right)(a(\Delta ABD))}$$

$$= \sqrt{\left(2\frac{BD}{AD}\right)\left(\frac{1}{2}(BD)(AD)\right)}$$

$$= \sqrt{(BD)^2}$$

$$= BD$$

The second recursive procedure can be expressed with the formula: $\left(\frac{2}{BD}\right)(a(\Delta ABD)) = AD$. It is used only once, but I call it recursive, because I imagine the scribe would have continued to employ it if he had finished the tablet. This procedure can be justified by:

$$\left(\frac{2}{BD}\right)\left(a(\Delta ABD)\right)$$
$$=\left(\frac{2}{BD}\right)\left(\frac{1}{2}(BD)(AD)\right)$$
$$=AD$$

Joseph asserts that the second equation was derived with the Pythagorean Theorem [**8**]. There is simply no evidence to support his claim. The reasoning presented above, although in modern terms, is fairly simple and not beyond the grasp of the Babylonians who could calculate areas of triangles. Joseph's claim that the Babylonians used similarity presents some difficulty. Is his interpretation a modern imposition? Since the Babylonians had no word for "angle" in their language, nor did they use parallelism, it seems strained to attribute to them a modern understanding of similar triangles.

But many historians of mathematics do believe the Babylonians understood similarity. Otto Neugebauer, one of the pioneers in interpreting ancient Babylonian mathematics, wrote in *The Exact Sciences in Antiquity* (1952), "The concept of similarity is utilized in numerous examples" [**9**]. Unfortunately, he did not elaborate. Jöran Friberg writes in "Methods and Traditions of Babylonian Mathematics," that the Babylonians must have had a similarity theorem along these lines: "The inclination, i.e., the ratio between the front and the flank, is the same for a given right triangle and for every subtriangle cut out of the given triangle by a perpendicular to either the front, the flank, or the diagonal of the given triangle" [**4**].

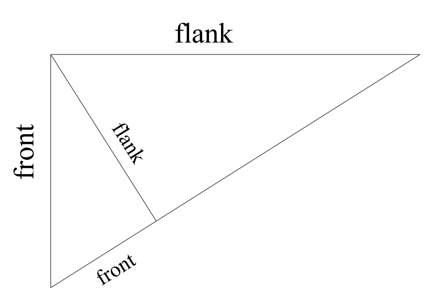


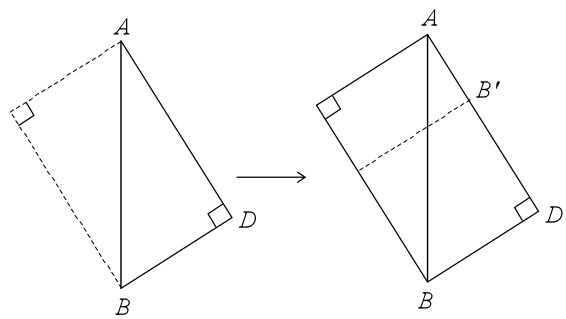
Diagram #5

In other words, the Babylonians may have developed a theorem along the lines of: for every right triangle formed within a larger right triangle, the legs of any subtriangles will be of the same ratio as the legs of the larger triangle, as illustrated in Diagram 5 above.

In the article "Mathematik," Friberg adds that while parallel lines and similar triangles are never explicitly mentioned, they exist in "a restricted sense," as the concepts of "equidistant lines" and "proportional triangles" are intuitively obvious [**5**]. His assertion that the Babylonians recognized that certain ratios of legs of right triangles were equal is strengthened by the fact that there exists a phrase for "the ratio of length to front." Friberg also wrote very emphatically in a response to my e-mail inquiry, "Similarity arguments are one of the cornerstones of Old Babylonian geometry" [**6**].

Jens Høyrup presents an interesting case for a Babylonian understanding of similarity in *Lengths, Widths, and Surfaces*. As noted when discussing his translation, Høyrup believes that all Babylonian algebra had a corresponding geometrical foundation. He writes that the scribe calculated the ratio between the two bases of ΔABC, *AB* and *AC*, and understood that the same ratio would apply to ΔABD. The scribe then multiplied $a(\Delta ABD)$ by twice this ratio $\left(2\dfrac{AB}{AC}\right)$ to produce a square of side *BD*. The scribe finds *BD* next, by taking the square root of the preceding expression [**7**].

Seeing the geometric argument helps one to understand Høyurp's conclusion. The scribe, he supposes, began with ΔABD. He multiplied the area of the triangle by two to produce the area of a rectangle. (*Diagram 5a)* He then multiplied the rectangle's area $(AD)(BD)$ by $\dfrac{AB}{AC}$ to produce the area of a square of side *BD*, since $\frac{AB}{AC} = \frac{BD}{AD}$. (*Diagram 5b)* He could take the square root of his result to find *BD* [**7**].



(a) (b)

Diagram #5a and 5b.

Høyrup notes subsequently that the same algorithm is used to find the bases of ΔADE. He argues that the use of the term, *gaba*, or "counterpart" to refer to the legs of the triangles shows that the Babylonians understood similarity on a basic level. Another set of tablets, BM 85194 #25-26, involving the construction of siege ramps, provides further evidence for Høyrup's claims. In BM 85194, the Babylonians, when given the area of a triangle, double it to produce a rectangle. They then scale it by the ratio of its sides to produce a square. By taking the square root, they can find one of the sides of the triangles [**7**]. This will be discussed later in more detail.

Indeed, because it involves an area multiplied by a ratio, $2\frac{AB}{AC}a(\Delta ABD) = BD^2$ seems likely to be an equation involving scaling. Once we have established the scaling method, as Høyrup suggests, it does not seem too large a stretch for them to realize that the legs of any inner right triangle are in the same ratio as the legs of the outer right triangle [**7**].

E.M. Bruins dissents from the idea that the Babylonians understood the concept of similar triangles. While Joseph writes that the Babylonians would have used similarity to conclude, in the diagram below, that $\frac{AB}{AC} = \frac{ED}{CE}$, Bruins believed that they would have begun from a different equation, $(AB)(CE) = (ED)(AC)$, which they would derive using the procedure below [**1**]:

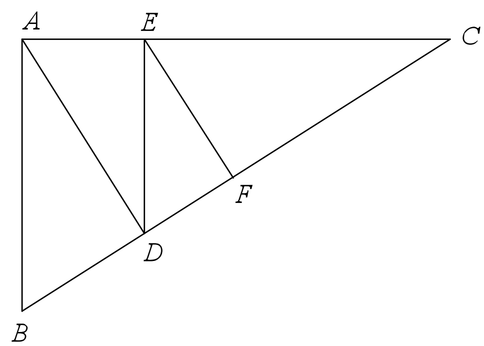


Diagram # 6

$$a(\Delta ABC) = a(\Delta CED) + a(EDBA)$$
$$\frac{1}{2}(AC)(AB) = \frac{1}{2}(CE)(ED) + \frac{1}{2}(AE)\left(ED + AB\right)$$
$$(AC)(AB) = (CE)(ED) + (AE)(ED + AB)$$
$$(AC)(AB) = (CE)(ED) + (AE)(ED) + (AE)(AB)$$
$$(AB)(AC - AE) = (ED)(CE + AE)$$
$$(AB)(CE) = (ED)(AC)$$

In a more general notation, the Babylonians may have realized that, given a right triangle and a right subtriangle, the longer base of the initial right triangle multiplied by the shorter base of the subtriangle equals the shorter base of the initial right triangle multiplied by the longer base of the subtriangle. We can easily see how the Babylonians could have used the above theorem to solve a problem that necessitates similar triangles in Joseph's rendition. Joseph's idea that the

Babylonians would have realized that $\frac{AB}{ED} = \frac{AC}{CE}$ and then cross-multiplied may be a modern imposition.

Bruins' argument encounters obstacles, however, because, this theorem can be proven only when the base of the subtriangle is parallel to the base of the larger triangle. Otherwise, let us return to Diagram 6 and this time deal with subtriangles created by drawing an altitude in the larger triangle, namely ΔABD and ΔACD.

$$a(\Delta ABC) = a(\Delta ABD) + a(ACD)$$
$$\frac{1}{2}(AB)(AC) = \frac{1}{2}(AD)(BD) + \frac{1}{2}(AD)(DC)$$
$$(AB)(AC) = (AD)(BD + DC)$$
$$(AB)(AC) = (AD)(BC)$$

In other words, the product of the shorter and longer legs of the initial triangle equals the product of the altitude of the initial triangle, which is also one of the bases of the subtriangle, and the hypotenuse.

But to reach the equation used in IM 55357, we would need to show that $(AB)(AD) = (BD)(AC)$ or that the product of the shorter leg of the initial triangle and the longer leg of the subtriangle equals the product of the longer leg of the initial triangle and the shorter leg of the subtriangle. We can remark only that if the Babylonians could have realized the conclusion for a subtriangle with a base parallel to the initial triangle, they would have also assumed it to hold true for the relationship between ΔABC and all other right subtriangles, ΔABD included. It seems as if this assumption would lack certainty.

An interesting question about the problem concerns the scribe's solution for *AD*. He uses $\left(\frac{2}{BD}\right)\left(a(\Delta ABD)\right) = AD$ to solve for *AD*. Why didn't he use the equation

$a(\Delta ABC) = \frac{1}{2}(BC)(AD)$ to solve for *AD*? If the scribe had been familiar with Bruins's method, it might have been more logical to use the above equation, as he would have understood what we demonstrated above: $(AB)(AC) = (AD)(BC)$. While Bruins's explanation of similarity may apply for other tablets, it does not seem to ring true in the context of this particular tablet.

I propose the following possibility: if the Babylonians had experimented with right triangles and right subtriangles, they may have noticed that the ratio between the areas of the two triangles equals the ratio between two equivalent sides squared. So, the Babylonians would have noted that:

$$\frac{a(\Delta ABD)}{a(\Delta ABC)} = \left(\frac{BD}{AB}\right)^2$$

$$\frac{a(\Delta ABD)}{\frac{1}{2}(AB)(AC)} = \frac{BD^2}{AB^2}$$

$$2\frac{AB}{AC}a(\Delta ABD) = BD^2$$

The assumption that the Babylonians would have noticed that the ratio of the sides squared is equal to the ratio of the areas of the triangles appears more complicated than that of Friberg's, that the ratio of the sides of similar triangles is equal. If the Babylonians did understand this theorem, how might they have derived it?

It seems likely that they might have asked themselves such questions as, "How many 3-4-5 right triangles could I fit into a 6-8-10 right triangle?" They may have initially supposed the answer to be two, because 6-8-10 is 3-4-5 doubled. But after drawing the diagram, they would find, to their surprise, that the answer was four. From there, they could experiment by fitting a variety of triangles into one another, eventually developing a general rule for the ratio between the area and the sides. The very existence of IM 55357 illustrates that the Babylonians must have experimented with areas of smaller triangles within larger ones. To create the area of ΔABD, it is likely that the scribe multiplied $a(\Delta ABC)$ by .36 or 0;21 36. For ΔADE, he multiplied ΔABC by .2304 or 0;13 49 26 24.

The theorem could have also been obtained in the following fashion:

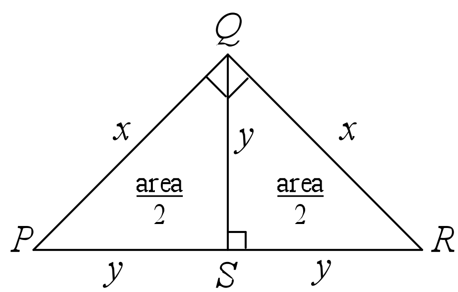


Diagram #7

Let ΔPQR be a right isosceles triangle. QS divides ΔPQR into an isosceles triangle, ΔPSQ. Let y be a leg of ΔPSQ. Let x be a leg of ΔPQR. It is then easy to develop the rule: $\frac{a(\Delta PSQ)}{a(\Delta PQR)} = \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2$. Is it possible that the Babylonians would have developed this theorem for more advanced cases than just isosceles triangles? Perhaps, but for the time being, we cannot produce their exact line of reasoning.

Friberg suggests in "Mathematik" that the Babylonians used similarity to derive something similar to the above result. If we return to the triangle of IM 55357, similarity shows us that:

$$a(\Delta ABC) = \frac{1}{2}(AB)(AC) = \left(\frac{AB}{AC}\right)\left(\frac{AC^2}{2}\right)$$

$$a(\Delta ABD) = \frac{1}{2}(AD)(AB) = \left(\frac{BD}{AD}\right)\left(\frac{AD^2}{2}\right) = \left(\frac{AB}{AC}\right)\left(\frac{AD^2}{2}\right).$$

From there, we can conclude that:

$$\frac{a(\Delta ABC)}{AC^2} = \frac{a(\Delta ABD)}{AD^2}.$$ [**5**]

Friberg's idea of a theorem relating the areas of the triangles to the sides squared adds yet one more to our wide variety of possibilities as to what extent the Babylonians understood similarity. At the very least, we know that, to use Friberg's terminology, they understood that the ratio of the legs of a right triangle is the same for any smaller right triangle cut out of it. But did they know more? Did they, as Høyrup suggests, perform operations involving scaling triangles to rectangles to squares?

To answer this question, it may be worthwhile to examine another tablet. **BM 85194 #26** may enable us to gain some more insight into the Babylonian perspective. BM 85194 presents the problem of a partially constructed siege ramp. Given the volume of a completed siege ramp and the dimensions of the ramp already constructed, how much farther does one need to extend the ramp horizontally to reach the city walls? It is extremely unlikely that this problem was used in a realistic context. Soldiers would not have known the volume of the unfinished ramp, nor would they stop building in order to measure how much farther to build, instead of calculating the geometry at the beginning. Furthermore, the ramp would not need to touch the edge of the wall exactly, but could exceed it slightly. Because of the unrealistic nature of this problem, we can conclude that it was used in a textbook or other pedagogical context.

Below, I present the diagram to which the tablet refers as well as a two-column explanation with Høyrup's translation on the left and a modern interpretation on the right [**7**].

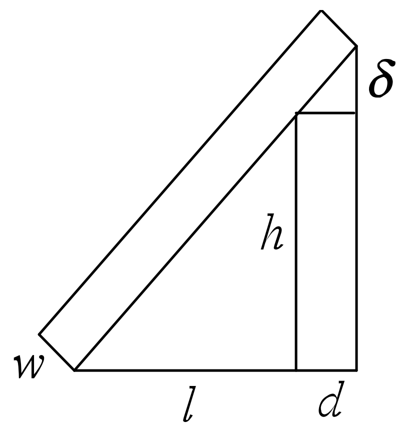


Diagram #8

Of earth, 1;30. A city inimical to Marduk I shall seize.

Away from the fundament of the earth a length of 0;32 in front of me I have gone.

0;36 the height of the earth. The length what I must stamp so that the city

I may seize? The length in front of the *hurhurum* (the vertical back front) reached so far what? You,

Igi 0;32 detach, 1;52 30 you see. 1;52 30 to 0;36, the height

Raise, 1;7 30 you see. Igi 6, the fundament of the earth, detach, 0;10 you see.

1;30, the earth, to 0;10 raise, 0;15 you see, 0;15 to 2 repeat.

0;30, you see, 0;30 to 1;7 30 raise, 0;33 45 you see

0;34 45, what is equalside? 0;45 is equalside, 0;45 the height of the city wall.

45, the height of the city wall, over 36, the height of the earth, what goes beyond? 0;09 it goes beyond.

The total volume (*V)* of the earth that the ramp will occupy is 1.5 cubic units.

The length(l) of the existing ramp is $\frac{8}{15}$

The height(h) of the existing ramp is $\frac{3}{5}$

Find the distance upward, δ , and across, d, that I would need to travel to seize the city.

Find the reciprocal of $\frac{8}{15} \rightarrow \frac{15}{8}$

$$\left(\frac{15}{8}\right)\left(\frac{3}{5}\right) = \frac{h}{l} = \frac{9}{8}; w = 6; \frac{1}{w} = \frac{1}{6}$$

$$\frac{V}{w} = \frac{1.5}{6} = .25 = Area(A);\ 2A = .5$$

$$2A\left(\frac{h}{l}\right) = \left(\frac{9}{8}\right).5 = .5625$$

$$\sqrt{2A\left(\frac{h}{l}\right)} = \sqrt{.5625} = .75 = (h+\delta)$$

$$(h+\delta) - \delta = .75 - .6 = .15 = \delta$$

Igi 1;7 30 detach, 0;53 20 you see, 0;53 20 to 0;09 raise.

$$\frac{1}{\frac{h}{l}} = \frac{1}{\frac{9}{8}} = \frac{8}{9} = \frac{l}{h}; \; \delta\frac{l}{h} = \frac{2}{15}$$

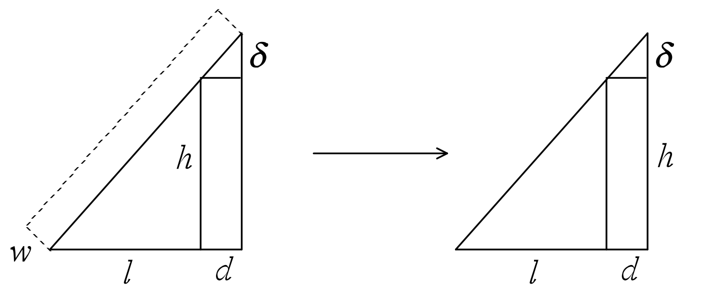
8 you see, 8 the length in front of you you stamp.

$$d = \frac{2}{15}$$

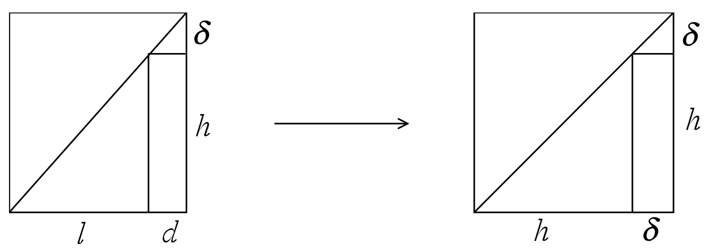
To analyze this procedure from a modern perspective, we need to look at the method represented by the formula, $\sqrt{2A\left(\frac{h}{l}\right)} = (h+\delta)$. Since $A = \frac{1}{2}(h+\delta)(l+d)$, we can rewrite the left side of the equation as $\sqrt{(h+\delta)(l+d)\left(\frac{h}{l}\right)}$. We notice that the triangles formed by *h* and *l* and formed by δ and *d* are similar. Therefore, $\frac{h}{l} = \frac{\delta}{d}$. From this equation, we can conclude that $\sqrt{(h+\delta)(l+d)\left(\frac{h}{l}\right)} = \sqrt{(h+\delta)\left(l\frac{h}{l} + d\frac{\delta}{d}\right)} = \sqrt{(h+\delta)^2}$. Knowing $h+\delta$, we can easily find δ. We then use $\delta\frac{l}{h} = d$ to solve for *d*. But we are faced with a recurring problem of interpretation: while this reasoning does provide a justification for the method used in the tablet, we have no way of knowing whether it was in fact the reasoning the Babylonians used in creating BM 85194 #26.

In *Lengths, Widths, and Surfaces*, Høyrup presents a geometrical interpretation of the Babylonian method. He notes that the tablet begins by taking *l/h*, which he claims they treated as the scaling ratio that would make the length of the triangle equal the width, 9/8. Then, the scribe takes the volume of the entire space and divides by the width, *w*, to produce a two-dimensional triangle *(8a)*. The scribe doubles the area of the triangle to produce a rectangle *(8b)*. He then multiplies the area by *l/h* to scale *l* + *d* to $h+\delta$ and produce a square of side $h+\delta$. *(8b)*

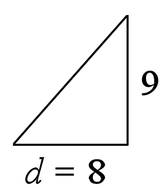
He then takes the square root of the area to find $h+\delta$. The scribe subtracts *h* to find δ. He then multiplies by *h/l* to reverse the operation performed on δ and find *d* [**7**].



(9a)



(9b)



(9c)

Diagram # 8

The question demanded by Høyrup's interpretation is: How does the scribe realize that multiplying by the area by $\frac{l}{h}$ transforms one side of the rectangle, $l+d$ to $h+\delta$? One possibility is that the Babylonians had developed similarity, as in the proof we examined above. Another is that they realized, perhaps because of some notion of slope, that the ratio of the two sides on the hypotenuse of any subtriangle will be the same as the ratio of sides of the original triangle. Thus, by finding the ratio of the sides of the partially-built ramp, the scribe could use that ratio to multiply by the area of the completed ramp and produce a square. It is difficult, though, to find a geometric image corresponding to the transformation of δ back into d by multiplying δ by $\frac{h}{l}$.

IM 55357 and BM 85194 #26 show us that the Babylonians had a limited understanding of similarity. At the very least, as per Friberg's claim, they understood that the ratio of the legs of any triangle and the ratio of the legs of any right subtriangle are equal. Høyrup notes that in both tablets, the Babylonians could scale one side of a two-dimensional rectangle to produce a square of proportional area. In BM 85194, the Babylonians seemed to have been scaling a scalene triangle to an isosceles triangle [**7**]. We know that the Babylonians developed other problems in which they scaled rectangles to squares of equal area [**1**]. There is still much left unknown, but it seems that in addition to scaling to equal area, the Babylonians also scaled rectangles to proportional area.

In any event, Euclid writes in Book VI of *Elements*, "In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another" [**3**]. It is highly probable that the Babylonians reached a similar claim seventeen centuries before, which is quite the testament to their civilization.

Appendix: What exactly is IM55357 solving for?

Taha Baqir, the former curator of the Iraqi Museum, who introduced the tablet in the 1950 edition of the journal *Sumer*, presents one opinion. He translates "width" as "perpendicular," and, hence, believes it refers to the three altitudes in the diagram, *AD, ED*, and *EF*. He believes "the upper length" to be *BD*, because *BD* is the first length the tablet finds [**2**]. He translates *ush-LUM*, Robson's "middle length" [**10**], as "segment" or "partial length" with the connotation of a length cut from another length. Since *DC* is cut from the hypotenuse, *BC*, and it is the next length found, he identifies *DC* as "the middle length." The "third length" he interprets as either of the two sides *AE* or *EC* [**2**].

Jöran Friberg adopted a different approach in his 1981 publication, "Methods and Traditions of Babylonian Mathematics." He writes that in asking for the upper, middle, and lower lengths, the problem requests each subtriangle's short leg, *BD*, *DF* and *AE*. Friberg does not comment as to the use of the word "vertical," but since he views the entire tablet as a recursive formula, he writes that "the problem consists in computing [all] the sides of the subtriangles" [**2**].

George Gheverghese Joseph believes that "length" makes reference each of the inscribed triangles' short leg, namely *BD*, *AE* and *DF*, while "width" indicates the altitude of the triangle, *AD* [**8**].

According to Jens Høyrup, the tablet uses the Babylonian word for "length," *ush*, in each of "the upper length," "the middle length," and "the lower length." He also uses *ush* to identify the longer leg of ΔABC, *AC*. Therefore, it is likely that the scribe referred to the corresponding side in each of the subtriangles. In other words, he intended *AD*, *DE* and *EF* to be the upper, middle and lower lengths, respectively. Høyrup translates “the vertical” as “the descendent,” and he believes that it refers to one of the sides of the remaining triangle, *EC* or *FC* [**7**].

References

1. Barry, Donald, Unpublished book on the early history of the Pythagorean theorem in various civilizations.
2. Baqir, T, Another important mathematical text from Tell Harmal, *Sumer* **6** (1950), 130-148.
3. Euclid, *The Elements*, Ed. H.S. Hall and F.H. Stevens. The Macmillan Company, London, 1904, pp344.
4. Friberg, Jöran, Methods and Traditions in Babylonian Mathematics, *Historia Mathematica* **8** (1981): 311-315.
5. Friberg, Jöran, “Mathematik,” *Reallexikon der Assyriologie und Vorderasiatischen Archäologie* (1987): 559-560.
6. Friberg, Jöran, Personal correspondence, dated 15 November 2005.
7. Høyrup, Jens, *Lengths, Widths, and Surfaces, A Portrait of Old Babylonian Algebra and its Kin*. Springer-Verlag, New York, 2002, pp. 217-223; 231-234
8. Joseph, George Gheverghese, *The Crest of the Peacock: Non-European Roots of Mathematics*, Princeton University Press, 2000.
9. Neugebauer, Otto, *The Exact Sciences in Antiquity*, Princeton University Press, 1952, pp.42.
10. Robson, Eleanor. Personal correspondence, dated 7 November 2005 and 1 January 2006.